Boundedness for Second Order Differential Equations with Jumping p-Laplacian and an oscillating term

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Abstract

In this paper, we are concerned with the boundedness of all the solutions for a kind of second order differential equations with p-Laplacian and an oscillating term $(\phi_p(x'))'+a\phi_p(x^+)-b\phi_p(x^-)=G_x(x,t)+f(t)$, where$x^+=\max(x,0)$,$x^-=\max(-x,0)$,$\phi_p(s)=|s|^{p-2}s$,$p\geq 2$, a and b are positive constants $(a\neq b)$,the perturbation $f(t)\in\mathcal{C}^{23}(\mathbb{R}/2\pi_p\mathbb{Z})$, the oscillating term $G\in\mathcal{C}^{21}(\mathbb{R}\times\mathbb{R}/2\pi_p\mathbb{Z})$,where $\pi_p=\frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin\frac{\pi}{p}}$, and $G(x,t)$ satisfies $|D_x^iD_t^jG(x,t)|\leq C,\quad 0\leq i+j\leq 21$, and $|D_t^j\hat{G}|\leq C,\quad 0\leq j\leq 21$ for some $C>0$, where $\hat{G}$ is some function satisfying $\frac{\partial\hat{G}}{\partial x}=G$.

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1 Introduction

One of the most studied semilinear Duffing's equations is

$$x''+ax^+-bx^-=f(x,t), \tag{1.1}$$

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where $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, $f(x,t)$ is a smooth 2π-periodic function on t, a and b are positive constants ($a \neq b$).

If $f(x,t)$ depends only on t, the equation (1.1) becomes

$$x'' + ax^+ - bx^- = f(t), \quad f(t + 2\pi) = f(t), \tag{1.2}$$

which had been studied by Fucik [6] and Dancer [3] in their investigations of boundary value problems associated to equations with "jumping nonlinearities". For recent developments, we refer to [7, 8, 11] and references therein.

In 1996, Ortega [20] proved the Lagrangian stability for the equation

$$x'' + ax^+ - bx^- = 1 + \gamma h(t) \tag{1.3}$$

if $|\gamma|$ is sufficiently small and $h \in \mathcal{C}^4(\mathbb{S}^1)$.

On the other hand, when $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q}$, Alonso and Ortega [2] proved that there is a 2π-periodic function $f(t)$ such that all the solutions of Eq. (1.2) with large initial conditions are unbounded. Moreover for such a $f(t)$, Eq. (1.2) has periodic solutions.

In 1999, Liu [16] removed the smallness assumption on $|\gamma|$ in Eq. (1.3) when $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q}$ and obtained the same result.

For the more general equation

$$x'' + ax^+ - bx^- + \phi(x) = e(t) \tag{1.4}$$

Wang [23] and Wang [24] considered the Lagrangian stability when the perturbation $\phi(x)$ is bounded.And Yuan [25] investigated the existence of quasiperiodic solutions and Lagrangian stability when $\phi(x)$ is unbounded.

Fabry and Mawhin [5] investigated the equation

$$x'' + ax^+ - bx^- = f(x) + g(x) + e(t) \tag{1.5}$$

under some appropriate conditions, they get the boundedness of all solutions.

Yang [27] considered more complicated nonlinear equation with p-Laplacian operator

$$((\phi_p(x'))' + (p-1)[a\phi_p(x^+) - b\phi_p(x^-)] + f(x) + g(x) = e(t). \tag{1.6}$$

Using Moser's small twist theorem, he proved that all the solutions are bounded,when $\frac{1}{a^{\frac{1}{p}}} + \frac{1}{b^{\frac{1}{p}}} = \frac{2m}{n}$, $m, n \in \mathbb{N}$, the perturbation $f(x)$ and the oscillating term g are bounded. For the case when $\frac{1}{a^{\frac{1}{p}}} + \frac{1}{b^{\frac{1}{p}}} = 2\omega^{-1}$, where $\omega \in \mathbb{R}^+ \backslash \mathbb{Q}$, the perturbation $f(x)$ is bounded, Yang [26] studied the following equation

$$(\phi_p(x'))' + a\phi_p(x^+) - b\phi_p(x^-) + f(x) = e(t). \tag{1.7}$$

and came to the conclusion that every solution of the equation is bounded.

In 2004, Liu [17] studied equation

$$(\phi_p(x'))' + a\phi_p(x^+) - b\phi_p(x^-) = f(x,t), f(x,t+2\pi) = f(x,t) \tag{1.8}$$

where $p > 1$, for the cases when $\frac{\pi_p}{a^{\frac{1}{p}}} + \frac{\pi_p}{b^{\frac{1}{p}}} = \frac{2\pi}{n}$ and $f \in \mathcal{C}^{(7,6)}(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})$ and satisfies that

(i) the following limits exists uniformly in t

$$\lim_{x\to\infty} f(x,t) = f_{\pm}(t)$$

(ii) the following limits exists uniformly in t

$$\lim_{x\to\infty} x^m \frac{\partial^{m+n}}{\partial x^m \partial t^n} f(x,t) = f_{\pm,m,n}(t)$$

for $(n,m) = (0,6),\ (7,0)$ and $(7,6)$. Moveover, $f_{\pm,m,n}(t) \equiv 0$ for $m = 6,\ n = 0,7$. He comes to the conclusion that all solutions are bounded and the existence of quasi-periodic solutions.

In 2012,Jiao,Piao and Wang [9] considered the bounededness of equations

$$x'' + \omega^2 x + \phi(x) = G_x(x,t) + f(t), \tag{1.9}$$

and

$$x'' + ax^+ - bx^- = G_x(x,t) + f(t). \tag{1.10}$$

Inspired by the above references, we are going to study the boundedness of all solutions for the more general equation

$$(\phi_p(x'))' + a\phi_p(x^+) - b\phi_p(x^-) = G_x(x,t) + f(t) \tag{1.11}$$

Our main results are as follows:

Theorem 1 *Assume $f(t) \in \mathcal{C}^{23}(\mathbb{R}/2\pi_p\mathbb{Z})$, $G \in \mathcal{C}^{21}(\mathbb{R} \times \mathbb{R}/2\pi_p\mathbb{Z})$,and $G(x,t)$ satisfies*

$$|D_x^i D_t^j G(x,t)| \leq C, \quad 0 \leq i+j \leq 21 \tag{1.12}$$

and

$$|D_t^j \hat{G}| \leq C, \quad 0 \leq j \leq 21 \tag{1.13}$$

for some $C > 0$, where $\hat{G}$ is some function satisfying $\frac{\partial \hat{G}}{\partial x} = G$,and $\omega = \frac{1}{2}(\frac{1}{a^{\frac{1}{p}}} + \frac{1}{b^{\frac{1}{p}}})$, and $\omega \in \mathbb{R}^+ \backslash \mathbb{Q}$ satisfy the Diophantine condition:

$$|m\omega + n| \geq \frac{\gamma}{|m|^\tau}, \quad \forall \ \ (m,n) \neq (0,0) \in \mathbb{Z}^2, \tag{1.14}$$

where $1 < \tau < 2,\ \gamma > 0$, and $[f] = \frac{1}{2\pi_p}\int_0^{2\pi_p} f(t)dt \neq 0$, where $\pi_p = \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin\frac{\pi}{p}}$. Then equation (1.11) possesses Lagrange stability, i.e. if $x(t)$ is any solution of equation (1.11), then it exists for all $t \in R$ and $\sup_{t\in R}(|x(t)| + |\dot{x}(t)|) < \infty$.

Remark 1.1 *In the above, γ can be any positive number. Thus our statement holds true for ω of full measure.*

Remark 1.2 *In Liu[17], it is required that f satisfies the limit condition, which is not satisfied by the function G in our situation. Thus our situation is more general.*

The main idea is as follows: By means of transformation theory the original system outside of a large disc $D = \{(x, x') \in R^2 : x^2 + x'^2 \leq r^2\}$ in (x, x')-plane is transformed into a perturbation of an integrable Hamiltonian system. The Poincaré map of the transformed system is closed to a so-called twist map in $R^2 \backslash D$. Then Moser's twist theorem guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the (x, x')-plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x, x', t) \in R^2 \times R$, which confines the solutions in the interior and which leads to a bound of these solutions.

The remain part of this paper is organized as follows. In section 2, we introduce action-angle variables and exchange the role of time and angle variables. In section 3, we construct canonical transformations such that the new Hamiltonian system is closed to an integrable one. In section 4, we will prove the Theorem 1 by Moser's twist theorem.

Throughout this paper, $F(x) = \int_0^x f(s)ds, F(0) = 0$, c and C are some positive constants without concerning their quantity.

2 Some Canonical transformations

In this section, we will state some technical lemmas which will be used in the proof of Theorem 1. Throughout this section, we assume the hypotheses of Theorem 1 hold.

2.1 Action-angle variables

Borrowing the idea from Liu [17] and Yang [26],we introduce a new variables y as $y = -\varphi_p(\omega x)$, let q be the conjugate exponent of $p : p^{-1} + q^{-1} = 1$. Then (1.11) is changed into the form

$$x' = -\omega^{-1}\varphi_q(y), y' = \omega^{-1}[a_1\varphi_p(x^+) - b_1\varphi_p(x^-)] - \omega^{p-1}[G_x(x,t) + f(t)] \tag{2.1}$$

where $a = \omega^{-p}a_1$, $b = \omega^{-p}b_1$ and a_1, b_1 satisfy

$$a_1^{-\frac{1}{p}} + b_1^{-\frac{1}{p}} = 2, \tag{2.2}$$

which is a planar non-autonomous Hamiltonian system

$$x' = -\frac{\partial H}{\partial y}(x, y, t), y' = \frac{\partial H}{\partial y}(x, y, t) \tag{2.3}$$

where

$$H(x,y,t)=\frac{\omega^{-1}}{q}|y|^q+\frac{\omega^{-1}}{p}(a_1|x^+|^p+b_1|x^-|^p)-\omega^{p-1}[G(x,t)+f(t)x].$$

Let $C(t)=\sin_p t$ be the solution of the following initial value problem

$$(\varphi_p(C'(t)))'+\varphi_p(C(t))=0,\quad C(0)=0,C'(0)=1. \tag{2.4}$$

Then it follows from [16] that $C(t)=\sin_p(t)$ is a $2\pi_p$-period C^2 odd function with $\sin_p(\pi_p-t)=\sin_p(t)$, for $t\in[0,\frac{\pi_p}{2}]$ and $\sin_p(2\pi_p-t)=-\sin_p(t)$, for $t\in[\pi_p,2\pi_p]$. Moreover for $t\in(0,\frac{\pi_p}{2})$, $C(t)>0$, $C'(t)>0$, and $C:[0,\frac{\pi_p}{2}]\to[0,(p-1)^{\frac{1}{p}}]$ can be implicitly given by

$$\int_0^{\sin_p t}\frac{ds}{(1-\frac{s^p}{p-1})^{\frac{1}{p}}}=t.$$

Lemma 2.1 *For $p\ge 2$ and for any $(x_0,y_0)\in R^2$, $t_0\in R$, the solution*

$$z(t)=(x(t,t_0,x_0,y_0),y(t,t_0,x_0,y_0))$$

of (2.1) satisfying the initial condition $z(t_0)=(x_0,y_0)$ is unique and exists on the whole t-axis.

The proof of uniqueness can be obtained similarly as the proof of Proposition 2 in [17], the global existence result can be proved similarly as Lemma 3.1 in [10]. Consider an auxiliary equation

$$(\phi_p(x'))'+a_1\phi_p(x^+)-b_1\phi_p(x^-)=0$$

Let $v(t)$ be the solution with initial condition: $(v(0),v'(0))=((p-1)^{\frac{1}{p}},0)$. Setting $\phi_p(v')=u$, then (v,u) is a solution of the following planar system:

$$x'=\phi_q(y),\quad y'=-a_1\phi_p(x^+)+b_1\phi_p(x^-)$$

where $q=p/(p-1)>1$. It is not difficult to prove that:
(i) $q^{-1}|u|^q+p^{-1}(a_1|v^+|^p+b_1|v^-|^p)\equiv\frac{a_1}{q}$;
(ii) $v(t)$ and $u(t)$ are $2\pi_p$-periodic functions.
(iii)$v(t)$ can be given by

$$v(t)=\begin{cases}\sin_p(a_1^{\frac{1}{p}}t+\frac{\pi_p}{2}), & 0\le t\le\frac{\pi_p}{2a_1^{\frac{1}{p}}},\\ -(\frac{a_1}{b_1})^{\frac{1}{p}}\sin_p b_1^{\frac{1}{p}}(t-\frac{\pi_p}{2a_1^{\frac{1}{p}}}), & \frac{\pi_p}{2a_1^{\frac{1}{p}}}<t\le\pi_p.\end{cases} \tag{2.5}$$

$$v(2\pi_p-t)=v(t),t\in[\pi_p,2\pi_p]. \tag{2.6}$$

Lemma 2.2 *Let* $I_p = \int_0^{\frac{\pi_p}{2}} \sin_p t dt$. *Then*

$$I_p = \frac{(p-1)^{\frac{2}{p}}}{p} B(\frac{2}{p}, 1-\frac{1}{p}),$$

where $B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}dt$ *for* $r>0, s>0$.

From the expression of $v(t)$ in (2.5), we obtain

$$\int_0^{\frac{\pi_p}{2a_1^{\frac{1}{p}}}} v(t)dt = \frac{I_p}{a_1^{\frac{1}{p}}}, \tag{2.7}$$

$$\int_{\frac{\pi_p}{2a_1^{\frac{1}{p}}}}^{\pi_p} v(t)dt = -\frac{a_1^{\frac{1}{p}} I_p}{b_1^{\frac{2}{p}}}. \tag{2.8}$$

This method has been used in [8].

We introduce the action and angle variables via the solution $(v(t), u(t))$ as follows.

$$x = d^{\frac{1}{p}} r^{\frac{1}{p}} v(\theta), y = d^{\frac{1}{q}} r^{\frac{1}{q}} u(\theta)$$

where $d = pa_1^{-1}$. This transformation is called a generalized symplectic transformation as its Jacobian is 1. Under this transformation, the system (2.1) is changed to

$$\theta' = \frac{\partial h}{\partial r}(r,\theta,t), r' = -\frac{\partial h}{\partial \theta}(r,\theta,t) \tag{2.9}$$

with the Hamiltonian function

$$h(r,\theta,t) = \omega^{-1} r - f_1(r,\theta,t) - \omega^{p-1} d^{\frac{1}{p}} r^{\frac{1}{p}} v(\theta) f(t) \tag{2.10}$$

where $f_1(r,\theta,t) = \omega^{p-1} G(d^{\frac{1}{p}} r^{\frac{1}{p}} v(\theta), t)$.

For any function $f(\cdot,\theta)$, we denote by $[f](\cdot)$ the average value of $f(\cdot,\theta)$ over $\mathbb{S}_p \triangleq \mathbb{R}/2\pi_p\mathbb{Z}$, that is,

$$[f](\cdot) := \frac{1}{2\pi_p} \int_0^{2\pi_p} f(\cdot,\theta) d\theta.$$

For the above function $f_1(r,\theta,t)$ in (2.10) we have

Lemma 2.3 *The following conclusion holds true:*

$$|D_r^i D_t^j f_1(r,\theta,t)| \le C \cdot r^{-\frac{i}{q}}, \quad 0 \le i+j \le 21. \tag{2.11}$$

Proof. The proof of this lemma can get directly from the definition of f_1 and the conditions in Theorem 1.
The following technique lemma will be used to refine the estimates on $[f_1](r,t)$.

Lemma 2.4 *Assume $f \in \mathcal{C}^1(\mathbb{R}/2\pi_p\mathbb{Z})$, $G(x,t) \in \mathcal{C}^1(\mathbb{R}^1 \times \mathbb{R}/2\pi_p\mathbb{Z})$ and $G'_x(x,t) = g(x,t)$. Suppose there are two positive constants $\bar{G}$ and $\bar{g}$ such that $|G(x,t)| \le \bar{G}$, $|g(x,t)| \le \bar{g}$ for any (x,t). Let $A(r,\theta) \in \mathcal{C}^2(\mathbb{R}^1 \times \mathbb{R}/2\pi_p\mathbb{Z})$ be of the form $A(r,\theta) = (r + h(r,\theta))^{\frac{1}{p}}$ with*

$$h, \frac{\partial h}{\partial \theta}, \frac{\partial^2 h}{\partial \theta^2} = O(r^{\frac{1}{p}}) \tag{2.12}$$

for $r \gg 1$.

Then for any constant $\delta_0 \in (0, \frac{1}{10})$ it holds that

$$\left| \int_0^{2\pi_p} f(\theta) g(Av(\theta), t) d\theta \right| \le C \cdot r^{-\delta_0}, \quad r \gg 1, \tag{2.13}$$

where C depends only on $\bar{G}$, $\bar{g}$ and $\|f\|_{C^0}$.

Proof. Let $[0, 2\pi_p] = I_1 \bigcup I_2$, where $I_1 = [0, r^{-2\delta_0}] \bigcup [\pi_p - r^{-2\delta_0}, \pi_p + r^{-2\delta_0}] \bigcup [2\pi_p - r^{-2\delta_0}, 2\pi_p]$ and $I_2 = [r^{-2\delta_0}, \pi_p - r^{-2\delta_0}] \bigcup [\pi_p + r^{-2\delta_0}, 2\pi_p - r^{-2\delta_0}]$. Then

$$\int_0^{2\pi} f(\theta) g(Av(\theta), t) d\theta = \int_{I_1} f(\theta) g(Av(\theta), t) d\theta + \int_{I_2} f(\theta) g(Av(\theta), t) d\theta.$$

Obviously, $|I_1| \le C \cdot r^{-2\delta_0}$, where $|\cdot|$ denotes the Lesbegue measure. Then from the boundedness of $g(x,t)$, it is easy to see that

$$\left| \int_{I_1} f(\theta) g(Av(\theta), t) d\theta \right| \le C \cdot r^{-2\delta_0}.$$

To estimate the integral on I_2, we first estimate the integral on the interval $I_{21} = [r^{-2\delta_0}, \pi_p - r^{-2\delta_0}]$.

Consider $D_\theta(Av(\theta)) = A'_\theta v(\theta) - Av'(\theta)$. From (2.12), it holds that $|Av'(\theta)| \ge c \cdot r^{\frac{1}{p} - 2\delta_0}$ and $A'_\theta \cdot v(\theta) = O(1)$ for $\theta \in I_{21}$, which implies

$$|D_\theta(Av(\theta))| \ge c \cdot r^{\frac{1}{p} - 2\delta_0}. \tag{2.14}$$

Similarly from the definition of A and the condition (2.12), we have

$$D_\theta^2(Av(\theta)) = D_\theta^2 A \cdot v(\theta) + 2D_\theta A \cdot v'(\theta) + Av''(\theta) = O(r^{\frac{1}{p}}). \tag{2.15}$$

By direct computation, we have

$$D_\theta(f(\theta)(D_\theta(Av(\theta)))^{-1}) = f' \cdot (D_\theta(Av(\theta)))^{-1} + f \cdot (D_\theta(Av(\theta)))^{-2} \cdot (-D_\theta^2(Av(\theta))).$$

Thus from (2.14) and (2.15), we obtain the estimate

$$|D_\theta(f(\theta)(D_\theta(Av(\theta)))^{-1})| \le C \cdot r^{4\delta - \frac{1}{p}}. \tag{2.16}$$

By integration by parts, we have that

$$\int_{I_{21}} f(\theta) g(Av(\theta), t) d\theta = \int_{I_{21}} f(\theta)(D_\theta(Av(\theta)))^{-1} dG(Av(\theta), t)$$

$$= \quad (D_\theta(Av(\theta)))^{-1} f(\theta) G(Av(\theta), t)|_{r^{-2\delta_0}}^{\pi_p - r^{-2\delta_0}} - \int_{I_{21}} G(Av(\theta), t) D_\theta(f(\theta) D_\theta((Av(\theta)))^{-1}) d\theta.$$

From (2.14) and (2.16) , for $\theta \in I_{21}$ it holds that

$$\left|(D_\theta(Av(\theta)))^{-1} f(\theta) G(Av(\theta), t)|_{\theta = r^{-2\delta_0}}\right|, \ \left|(D_\theta(Av(\theta)))^{-1} f(\theta) G(Av(\theta), t)|_{\theta = \pi_p - r^{-2\delta_0}}\right| \le C \cdot r^{4\delta_0 - \frac{1}{p}}$$

and

$$\left|G(Av(\theta), t) \cdot D_\theta(f(\theta) D_\theta((Av(\theta)))^{-1})\right| \le C \cdot r^{4\delta_0 - \frac{1}{p}}.$$

Similarly, we can have the same estimate for the other parts of I_2.

Hence from the fact $0 < \delta_0 < \frac{1}{10}$, we obtain (2.7). The proof of this lemma is completed. □

For $[f_1](r, t)$, we have the following result:

Corollary 2.1 *The following conclusion holds true:*

$$|D_r^i D_t^j [f_1](r, t)| \le C \cdot r^{-\delta_1 - \frac{i}{p}}, \quad 0 \le i + j \le 21, \tag{2.17}$$

where the constant δ_1 *is in* $(0, \frac{1}{10})$.

Proof. From the definition of f_1, we have $[f_1](r, t) = \frac{1}{2\pi_p} \int_0^{2\pi_p} G(r^{\frac{1}{p}} v(\theta), t) d\theta$. From (1.12) and (1.13), we know that G and $\hat{G}$ are bounded. Thus for $i + j = 0$, (2.17) is deduced from lemma 2.2 where we set $f \equiv 1$ and $A(r, \theta) = r^{\frac{1}{p}}$. For $i + j \ge 1$, it can be easily seen that $\frac{\partial^{i+j}}{\partial r^i \partial t^j} G$ are the sum of the term like

$$\frac{\partial^{k+j}}{\partial x^k \partial t^j} G(r^{\frac{1}{p}} v(\theta), t)(r^{\frac{1}{p}})^{(i_1)} \cdots (r^{\frac{1}{p}})^{(i_k)} \cdot (v(\theta))^k,$$

where $i_1 + \cdots i_k = i$. Thus (2.17) is implied from lemma 2.2 for the function $\frac{\partial^{k+j}}{\partial x^k \partial t^j} G(r^{\frac{1}{p}} v(\theta), t)$ and (1.12). This ends the proof of the lemma. □

2.2 Exchange of the roles of time and angle variables

According to Levi [12],the equality

$$rd\theta - hdt = -(hdt - rd\theta),$$

means if we can solve $r = r(h, t, \theta)$ from Eq.(2.9) as a function of h, t and θ, then we have

$$\frac{dh}{d\theta} = -\frac{\partial r}{\partial t}(h, t, \theta), \quad \frac{dt}{d\theta} = \frac{\partial r}{\partial h}(h, t, \theta), \tag{2.18}$$

i.e., Eq.(2.18) is a Hamiltonian system with Hamiltonian function $r = r(h, t, \theta)$ and now the action, angle and time variables are h, t, and θ, respectively.

From Eq.(2.10) and lemmas, it follows that

$$\lim_{r \to +\infty} \frac{h}{r} = \omega^{-1} > 0$$

and for $r \gg 1$

$$\frac{\partial h}{\partial r} = \omega^{-1} - \frac{\partial}{\partial r} f_1(r, \theta) - \frac{1}{p} f(t) \omega^{p-1} d^{\frac{1}{p}} r^{\frac{1}{p}-1} v(\theta) > 0.$$

By the implicit function theorem, we know that there is a function $R = R(h, t, \theta)$ such that

$$r(h, t, \theta) = \omega h - R(h, t, \theta). \tag{2.19}$$

Moreover, for $h \gg 1$,

$$|R(h, t, \theta)| \leq \omega h/2$$

and $R(h, t, \theta)$ is C^{19} in h and t.

From (2.10), it holds that

$$R = \omega f_1(\omega h - R, t, \theta) - \omega^p d^{\frac{1}{p}} (\omega h - R)^{\frac{1}{p}} v(\theta) f(t). \tag{2.20}$$

The proof of following two lemmas are slightly different to [15], here for the convenience of readers, we give the proofs of them.

Lemma 2.5 *Assume R is defined by (2.20) with $|R| \ll h$ for $h \gg 1$. Then it holds that*

$$|D_h^i D_t^j R| \leq C \cdot h^{n(i)}, \quad 0 \leq i + j \leq 21 \tag{2.21}$$

for $h \gg 1$, where $n(i) = -\frac{i}{q}$ for $i \geq 1$ and $n(0) = \frac{1}{p}$.

Proof. (i) $i + j = 0$. The proof for this case can be easily obtained from lemma 2.3 and the conditions in the Theorem .
(ii) $i + j = 1$. It is clear that for $h \gg 1$,

$$|\omega \frac{\partial f_1}{\partial r}(\omega h - R, t, \theta)| + |\frac{\omega^p}{p} d^{\frac{1}{p}} (\omega h - R)^{-\frac{1}{q}} v(\theta) f(t)| \leq \frac{1}{2}.$$

Define

$$\Delta(h,t,\theta) = 1 + \omega\frac{\partial f_1}{\partial r}(\omega h - R, t, \theta) - \frac{\omega^p}{p} d^{\frac{1}{p}}(\omega h - R)^{-\frac{1}{q}} v(\theta) f(t),$$

$$g_1 = \omega^2\frac{\partial f_1}{\partial r}(\omega h - R, t, \theta) - \frac{\omega^{p+1}}{p} d^{\frac{1}{p}}(\omega h - R)^{-\frac{1}{q}} v(\theta) f(t),$$

$$g_2 = -\omega^p d^{\frac{1}{p}}(\omega h - R)^{\frac{1}{p}} v(\theta) f(t) + \omega\frac{\partial f_1}{\partial t}(\omega h - R, t, \theta).$$

Then it follows that

$$\Delta \cdot \frac{\partial R}{\partial h} = g_1, \quad \Delta \cdot \frac{\partial R}{\partial t} = g_2. \tag{2.22}$$

From lemma 2.3,$p \geq 2$ and the boundedness of $f(t)$, we have $|g_1| \leq C \cdot h^{-\frac{1}{q}}$ and $|g_2| \leq C \cdot h^{\frac{1}{p}}$. Thus the proof for this case is completed.

(iii) $i + j = 2$. Lemma 2.3 implies that

$$|\frac{\partial \Delta}{\partial t}| \leq C \cdot h^{-\frac{1}{q}},\ |\frac{\partial \Delta}{\partial h}| \leq C \cdot h^{-\frac{2}{q}},\ |\frac{\partial g_1}{\partial t}| \leq C \cdot h^{-\frac{1}{q}},\ |\frac{\partial g_1}{\partial h}| \leq C \cdot h^{-\frac{2}{q}},\ |\frac{\partial g_2}{\partial h}| \leq C \cdot h^{-\frac{1}{q}},\ |\frac{\partial g_2}{\partial t}| \leq C \cdot h^{\frac{1}{p}}.$$

From the second equation of (2.22), we obtain

$$\Delta\frac{\partial^2 R}{\partial t^2} + \frac{\partial \Delta}{\partial t} \cdot \frac{\partial R}{\partial t} = \frac{\partial g_2}{\partial t}$$

and

$$\Delta\frac{\partial^2 R}{\partial t \partial h} + \frac{\partial \Delta}{\partial h} \cdot \frac{\partial R}{\partial t} = \frac{\partial g_2}{\partial h}.$$

The above inequalities and equations imply that

$$|\frac{\partial^2 R}{\partial t^2}| \leq C \cdot h^{\frac{1}{p}}, \quad |\frac{\partial^2 R}{\partial h \partial t}| \leq C \cdot h^{-\frac{1}{q}}.$$

From the first equation of (2.22), we know that

$$\Delta\frac{\partial^2 R}{\partial h^2} + \frac{\partial \Delta}{\partial h} \cdot \frac{\partial R}{\partial h} = \frac{\partial g_1}{\partial h},$$

which implies $|\frac{\partial^2 R}{\partial h^2}| \leq C \cdot h^{-\frac{2}{q}}$. Thus we complete the proof for this case.

In general, if

$$|D_h^i D_t^j R| \leq C \cdot h^{n(i)}, \quad 0 \leq i + j \leq m,$$

then it holds that

$$|D_h^i D_t^j \Delta| \leq C \cdot h^{-\frac{1}{q} + n(i)}, \quad |D_h^i D_t^j g_1| \leq C \cdot h^{-\frac{1}{q} - \frac{i}{q}}, \quad |D_h^i D_t^j g_2| \leq C \cdot h^{-\frac{i}{q}}.$$

Consequently, we obtain

$$|D_h^i D_t^j R| \leq C \cdot h^{n(i)}, \quad 0 \leq i + j \leq m + 1.$$

The proof is completed. □

In (2.20), we denote $R = -\omega^p d^{\frac{1}{p}}(\omega h)^{\frac{1}{p}} v(\theta) f(t) - R_1(h, t, \theta)$. Then

$$R_1 = \omega f_1(\omega h - R, t, \theta) - \frac{1}{p} \int_0^1 \omega^p d^{\frac{1}{p}} (\omega h - \tau R)^{-\frac{1}{q}} R v(\theta) f(t) d\tau. \tag{2.23}$$

Then we have the following conclusion:

Lemma 2.6 *It holds that*

$$|D_h^i D_t^j R_1| \leq C \cdot h^{-\frac{i}{q}}, \quad 0 \leq i + j \leq 21.$$

Proof. The lemma is easily followed from the following claim:

Claim

$$\begin{aligned} &|D_h^i D_t^j f_1(\omega h - \tau R, t, \theta)| \leq C \cdot h^{-\frac{i}{q}}, \\ &|D_h^i D_t^j (\omega h - \tau R)^{-\frac{1}{q}} d^{\frac{1}{p}} R v(\theta) f(t)| \leq C \cdot h^{-\frac{1}{q} - \frac{i}{q}} \end{aligned} \tag{2.24}$$

for $0 \leq i + j \leq 21$.

Proof of the claim. We only prove the first inequality of (2.24) and the proof for the other is similar.

(i) $i + j = 0$. The proof for this case can be obtained directly from lemma 2.1.

(ii) $i > 0, j = 0$. We have the following equality:

$$D_h^i f_1(\omega h - \tau R, t, \theta) = \sum \frac{\partial^k f_1}{\partial r^k}(u, t, \theta) \cdot \frac{\partial^{i_1} u}{\partial h^{i_1}} \cdots \frac{\partial^{i_k} u}{\partial h^{i_k}}$$

with $0 < k \leq i,\ i_1, \cdots, i_k > 0,\ i_1 + \cdots i_k = i$ and $u = \omega h - \tau R$. Assume there are $l (\leq k)$ numbers in $\{i_1, \cdots, i_k\}$ which is equal to 1. Then we obtain

$$|D_h^i f_1(u, t, \theta)| \leq C \cdot h^{-\frac{k}{q}} \cdot h^{-\frac{i_1 + \cdots i_k - l}{q}} \leq C \cdot h^{-\frac{i}{q}}.$$

(iii) $i = 0, j > 0$. By direct computation, we have

$$D_t^j f_1(\omega h - \tau R, t, \theta) = \sum \frac{\partial^{k+l} f_1}{\partial r^k \partial t^l}(u, t, \theta) \cdot \frac{\partial^{j_1} u}{\partial t^{j_1}} \cdots \frac{\partial^{j_k} u}{\partial t^{j_k}}$$

with $0 \leq k \leq j, 0 \leq l \leq j, k + l = j,\ j_1, \cdots, j_k > 0,\ j_1 + \cdots j_k = k$. It follows that

$$|D_t^j f_1(u, t, \theta)| \leq C \cdot h^{-\frac{k}{q}} \cdot h^{\frac{k}{p}} \leq C.$$

The last step,we get from that $p \geq 2$, $\frac{1}{p} + \frac{1}{q} = 1$,and $\frac{1}{p} \leq \frac{1}{q}$.

(iv) $i > 0, j > 0$. By direct computation, we have

$$D_h^i D_t^j \frac{\partial f_1}{\partial r}(u, \theta) = \sum \frac{\partial^{k_1 + k_2 + l} f_1}{\partial r^{k_1 + k_2} \partial t^l}(u, \theta) \cdot \frac{\partial^{i_1} u}{\partial h^{i_1}} \cdots \frac{\partial^{i_{k_1}} u}{\partial h^{i_{k_1}}} \cdot \frac{\partial^{l_1 + j_1} u}{\partial h^{l_1} \partial t^{j_1}} \cdots \frac{\partial^{l_{k_2} + j_{k_2}} u}{\partial h^{l_{k_2}} \partial t^{j_{k_2}}},$$

where $u = \omega h - \tau R$ and

$$0 \le k_1 \le i,\ 0 \le k_2 \le j,\ 0 \le l \le j,\ k_2 + l = j,\ i_1, \cdots, i_{k_1},\ j_1, \cdots, j_{k_2} > 0, \quad l_1, \cdots, l_{k_2} \ge 0,$$

$$i_1 + \cdots i_{k_1} + l_1 + \cdots + l_{k_2} = i, \quad j_1 + \cdots + j_{k_2} + l = j.$$

Assume that there are $m(\le k_1)$ numbers in $\{i_1, \cdots, i_{k_1}\}$ which is equal to 1. Then

$$|D_h^i D_t^j \frac{\partial f_1}{\partial r}| \le C \cdot h^{-\frac{k_1+k_2}{q}} \cdot h^{-\frac{i_1+\cdots+i_{k_1}+l_1+\cdots+l_{k_2}-m}{q}} \le C \cdot h^{-\frac{i}{q}}.$$

This ends the proof of the claim. □

From the definition of R_1, we can obtain the following conclusion:

Lemma 2.7 *For the function $[R_1](h,t)$, we have that*

$$|D_h^i D_t^j [R_1]| \le C \cdot (h^{-i} + h^{-\delta_1 - \frac{i}{q}}), \quad 0 \le i + j \le 21,$$

where $\delta_1 \in (0, \frac{1}{10})$.

From (2.19), (2.20) , we obtain that the Hamiltonian $r(h,t,\theta)$ in (2.19) is of the form:

$$r = \omega h + \omega^p d^{\frac{1}{p}} (\omega h)^{\frac{1}{p}} v(\theta) f(t) + R_1(h,t,\theta). \tag{2.25}$$

3 More canonical transformations

In this section, we will make some more canonical transformations such that the Poincaré map of the new system is close to twist map.

Lemma 3.1 *There exists a canonical transformation Φ_1 of the form:*

$$\Phi_1 : \quad \begin{cases} h &= \rho \\ t &= \tau + V_1(\rho,\tau,\theta) \end{cases}$$

where the functions V_1 are periodic in τ, θ. Under this transformation, the Hamiltonian system with Hamiltonian (2.25) is changed into the following one

$$\tilde{r} = \omega\rho + \omega^p d^{\frac{1}{p}} (\omega\rho)^{\frac{1}{p}} v(\theta)[f] + \tilde{R}_1(\rho,\tau,\theta), \tag{3.1}$$

Moreover, the new perturbation $\tilde{R}_1$ satisfies

$$|\frac{\partial^{i+j}}{\partial\rho^i \partial\tau^j} \tilde{R}_1| \le C \cdot \rho^{-\frac{i}{q}}, \quad 0 \le i + j \le 21. \tag{3.2}$$

Moreover, for the function$[\tilde{R}_1](\rho,\theta)$,it holds that

$$|D_\rho^i D_\tau^j [\tilde{R}_1]| \le C \cdot (\rho^{-i} + \rho^{-\delta_1 - \frac{i}{q}}). \quad 0 \le i + j \le 21. \tag{3.3}$$

Proof. We construct the canonical transformation by means of generating function:

$$\Phi_1: \quad h=\rho, \quad t=\tau+\frac{\partial S_1}{\partial \rho}(\rho,\tau,\theta).$$

Under this transformation, the new Hamiltonian function $\tilde{r}$ is of the form

$$\tilde{r}=\omega\rho+\omega^p d^{\frac{1}{p}}(\omega\rho)^{\frac{1}{p}}v(\theta)f(\tau+\frac{\partial S_1}{\partial \rho})+R_1(\rho,\tau+\frac{\partial S_1}{\partial \rho},\theta)+\frac{\partial S_1}{\partial \theta}$$

Let $S_1=-\int_0^\theta \omega^p d^{\frac{1}{p}}(\omega\rho)^{\frac{1}{p}}v(\vartheta)f(t)-[f]d\vartheta$,then we have

$$\tilde{r}(\rho,\tau,\theta)=\omega\rho+\omega^p d^{\frac{1}{p}}(\omega\rho)^{\frac{1}{p}}v(\theta)[f]+\tilde{R}_1(\rho,\tau,\theta)$$

where $\tilde{R}_1(\rho,\tau,\theta)=R_1(\rho,\tau+\frac{\partial S_1}{\partial\rho},\theta)=R_1(\rho,\tau,\theta)+\int_0^1\frac{\partial R_1}{\partial t}(\rho,\tau+s\frac{\partial S_1}{\partial\rho},\theta)\frac{\partial S_1}{\partial\rho}ds$ From 2.6 and the definition of $\tilde{R}_1$, we can get the estimates (3.2) ,(3.3) can get from 2.7 and the definition of $\tilde{R}_1$. □

Lemma 3.2 *There exists a canonical transformation Φ_2 of the form:*

$$\Phi_2: \quad \begin{cases} \rho &= I \\ \tau &= s+V_2(I,\theta) \end{cases}$$

with $\tilde{T}(I,\theta+2\pi_p)=\tilde{T}(I,\theta)$,such that the system with Hamiltonian (3.1) is transformed into the form:

$$\frac{\partial I}{\partial \theta}=-\frac{\partial \bar{r}}{\partial s}(I,s,\theta), \quad \frac{\partial s}{\partial \theta}=\frac{\partial \bar{r}}{\partial I}(I,s,\theta) \tag{3.4}$$

with $\bar{r}(I,s,\theta)=\omega I+c^*I^{\frac{1}{p}}+\tilde{R}_2(I,s,\theta)$ and $c^*\neq 0$, where we use the fact that $[f]\neq 0$. Moreover, the new perturbation $\tilde{R}_2$ satisfies

$$|D_I^i D_s^j \tilde{R}_2)|\leq C\cdot I^{-\frac{i}{q}}, \quad 0\leq i+j\leq 21. \tag{3.5}$$

Moreover, for the function $[\tilde{R}_2]_0(I)=(\frac{1}{2\pi_p})^2\int_0^{2\pi_p}\int_0^{2\pi_p}\tilde{R}_2(I,s,\theta)dsd\theta$, it holds that

$$|D_I^i[\tilde{R}_2]_0|\leq C\cdot(I^{-i}+I^{-\delta_1-\frac{i}{q}}), \quad 0\leq i\leq 21. \tag{3.6}$$

Proof. The proof is similar to [18], but for the convenience of readers we still give a detailed argument. We shall look for the required transformation Φ_2 by means of a generating function $S_2(I,s,\theta)$, so that Φ_2 is implicitly defined by

$$\Phi_2: \quad \rho=I+\frac{\partial}{\partial s}S_2(I,s,\theta), \quad \tau=s+\frac{\partial}{\partial I}S_2(I,s,\theta). \tag{3.7}$$

Under this transformation, the system is changed into the form:

$$\frac{\partial I}{\partial \theta} = -\frac{\partial \bar{r}}{\partial s}(I,s,\theta), \quad \frac{\partial s}{\partial \theta} = \frac{\partial \bar{r}}{\partial I}(I,s,\theta)$$

the new Hamiltonian function $\bar{r}$ is of the form

$$\bar{r} = \omega\rho + \omega^{p+\frac{1}{p}} d^{\frac{1}{p}}[f]\rho^{\frac{1}{p}} v(\theta) + \tilde{R}_1(\rho,\tau,\theta) + \frac{\partial S_2}{\partial \theta}$$

Now we choose

$$S_2 = -\int_0^\theta \omega^{p+\frac{1}{p}} d^{\frac{1}{p}}[f]\rho^{\frac{1}{p}} v(\vartheta) - c^*\rho^{\frac{1}{p}} d\vartheta$$

where $c^* = \omega^{p+\frac{1}{p}} d^{\frac{1}{p}}[f] \neq 0$. Obviously,$S_2$ does not depend on s and it is $2\pi_p-$periodic in θ. Hence $\rho = I$. Let

$$\tilde{T}(I,\theta) = \frac{\partial S_2}{\partial I}.$$

Then the canonical transformation Φ_2 is of the form

$$\rho = I, \tau = s + \tilde{T}(I,\theta).$$

Let

$$\tilde{R}_2(I,s,\theta) = \tilde{R}_1(\rho,s,\theta) + \int_0^1 \frac{\partial \tilde{R}_1}{\partial \tau}(\rho, s+m\tilde{T},\theta)\tilde{T}dm. \tag{3.8}$$

From (3.2) in lemma3.1,we can get (3.5) easily, and (3.6) can get from (3.3). The proof of this lemma is completed. □

For convenience, we denote

$$\bar{r} = \omega I + \bar{r}_1(I) + \bar{r}_2(I,s,\theta), \tag{3.9}$$

with $\bar{r}_1 = c^* I^{\frac{1}{p}}$, $\bar{r}_2(I,s,\theta) = \tilde{R}_2(I,s,\theta)$, then from the definition of $\bar{r}_1$,we can know that, $\bar{r}_1$ satisfying

$$c \cdot I^{\frac{1}{p}-i} \le |\bar{r}_1^{(i)}(I)| \le C \cdot I^{\frac{1}{p}-i}, \tag{3.10}$$

$\bar{r}_2$ have the same estimate with $\tilde{R}_2$ in lemma3.2,i.e.

$$|D_I^i D_s^j \bar{r}_2)| \le C \cdot I^{-\frac{i}{q}}, \quad 0 \le i+j \le 21. \tag{3.11}$$

$$|D_I^i [\bar{r}_2]_0| \le C \cdot (I^{-i} + I^{-\delta_1 - \frac{i}{q}}), \quad 0 \le i \le 21. \tag{3.12}$$

The following results are similarity to [9],here for the convenience of readers, we still give the proof of these lemmas.

Lemma 3.3 *Let $0 < \delta_1 < \frac{1}{10}$ be a constant. Consider the Hamiltonian*

$$\bar{r}(I, s, \theta) = \omega I + \bar{r}_1(I) + \mathscr{R}(I, s, \theta), \tag{3.13}$$

where $\mathscr{R}$ satisfies

$$|D_I^i D_s^j \mathscr{R}| \leq C \cdot I^{-\varepsilon - \frac{i}{q}} \tag{3.14}$$

for $0 \leq i + j \leq l$ with $\varepsilon \geq 0$.

Then there exists a canonical transformation Φ_3 of the form:

$$\Phi_3 : \quad \begin{cases} I &= \varrho + u_3(\varrho, \varsigma, \theta) \\ s &= \varsigma + v_3(\varrho, \varsigma, \theta) \end{cases}$$

such that the system with Hamiltonian (3.13) is transformed into the following one

$$\hat{r}(\varrho, \varsigma, \theta) = \omega \varrho + \hat{r}_1(\varrho) + \mathscr{R}_1(\varrho, \varsigma, \theta), \tag{3.15}$$

where $\hat{r}_1(\varrho) = \bar{r}_1(\varrho) + [\mathscr{R}]_0(\varrho)$ with $[\mathscr{R}]_0(\varrho) = (\frac{1}{2\pi_p})^2 \int_0^{2\pi_p}\int_0^{2\pi_p} \mathscr{R}(\varrho, \tau, \theta) d\tau d\theta$ and $\mathscr{R}_1$ satisfies

$$|D_\varrho^i D_\varsigma^j \mathscr{R}_1| \leq C \cdot \varrho^{-\varepsilon - \frac{1}{q} - \frac{i}{q}}, \quad 0 \leq i + j \leq l - 3. \tag{3.16}$$

Proof. We will prove this lemma by means of Principle Integral method instead of Fourier series method. Let Φ_3 be of the following form:

$$I = \varrho + \frac{\partial S_3}{\partial \tau}(\varrho, s, \theta), \quad \varsigma = s + \frac{\partial S_3}{\partial \varrho}(\varrho, s, \theta),$$

where the generating function $S_3(\varrho, s, \theta)$ satisfies $S_3(\varrho, s+2\pi_p, \theta) = S_3(\varrho, s, \theta+2\pi_p) = S_3(\varrho, s, \theta)$ and will be determined later.

Then the transformed Hamiltonian is

$$\begin{aligned} \hat{r} &= \omega(\varrho + \tfrac{\partial S_3}{\partial s}) + \bar{r}_1(\varrho + \tfrac{\partial S_3}{\partial s}) + \mathscr{R}(\varrho + \tfrac{\partial S_3}{\partial s}, s, \theta) + \tfrac{\partial S_3}{\partial \theta} \\ &= \omega \varrho + \bar{r}_1(\varrho) + [\mathscr{R}]_0(\varrho) + \omega \tfrac{\partial S_3}{\partial s} + \tfrac{\partial S_3}{\partial \theta} + R + \mathscr{R}_1, \end{aligned}$$

where

$$R = \mathscr{R}(\varrho, s, \theta) - [\mathscr{R}]_0(\varrho)$$

and

$$\mathscr{R}_1 = \int_0^1 \bar{r}_1'(\varrho + \lambda \frac{\partial S_3}{\partial s}) \frac{\partial S_3}{\partial s} d\lambda + \int_0^1 \frac{\partial \mathscr{R}}{\partial I}(\varrho + \lambda \frac{\partial S_3}{\partial s}, s, \theta) \frac{\partial S_3}{\partial s} d\lambda. \tag{3.17}$$

Obviously, it holds that

$$(\frac{1}{2\pi_p})^2 \int_0^{2\pi_p} \int_0^{2\pi_p} R(\varrho, s, \theta) ds d\theta = 0. \tag{3.18}$$

Now we determine the periodic function S_3 by the following equation

$$\omega\frac{\partial S_3}{\partial s}(\varrho,s,\theta)+\frac{\partial S_3}{\partial\theta}(\varrho,s,\theta)+R(\varrho,s,\theta)=0, \tag{3.19}$$

whose characteristic equation is

$$\frac{ds}{\omega}=\frac{d\theta}{1}=\frac{dS_3}{-R(\varrho,s,\theta)}.$$

Obviously, the characteristic equation possesses two independent Principle Integrals as follows:

$$s-\omega\theta=c_1$$

and

$$S_3+\int_0^\theta R(\varrho,s-\omega\theta+\omega\phi,\phi)d\phi=c_2.$$

Thus the solution of (3.19) is of the form:

$$S_3(\varrho,s,\theta)=-\int_0^\theta R(\varrho,s-\omega\theta+\omega\phi,\phi)d\phi+\Omega(\varrho,s-\omega\theta) \tag{3.20}$$

with Ω a differentiable function determined later.

To ensure S_3 be $2\pi_p$-periodic on s and θ, Ω must be $2\pi_p$-periodic on the second variable, that is $\Omega(\varrho,x+2\pi_p)=\Omega(\varrho,x)$. Then by direct computation, we obtain that S_3 is $2\omega\pi_p$-periodic on s.

Next we determine Ω by the periodicity of S_3 on θ.

Let $J(\varrho,x)=-\int_0^{2\pi_p}R(\varrho,x+\omega\phi,\phi)d\phi$. Then we have

$$\begin{aligned}&S_3(\varrho,s,\theta+2\pi_p)=-\textstyle\int_0^{\theta+2\pi_p}R(\varrho,s-\omega(\theta+2\pi_p-\phi),\phi)d\phi+\Omega(\varrho,s-\omega(\theta+2\pi_p))\\&=J(\varrho,s-\omega(\theta+2\pi_P))-\textstyle\int_{2\pi_p}^{2\pi_p+\theta}R(\varrho,s-\omega(\theta+2\pi_p-\phi),\phi)d\phi+\Omega(\varrho,s-\omega(\theta+2\pi_p)).\end{aligned}$$

On the other hand, from $R(\varrho,s,\phi+2\pi_p)=R(\varrho,s,\phi)$ we have

$$\int_{2\pi_p}^{2\pi_p+\theta}R(\varrho,s-\omega(\theta+2\pi_p-\phi),\phi)d\phi=\int_0^\theta R(\varrho,s-\omega(\theta-\phi),\phi)d\phi,$$

which implies that

$$S_3(\varrho,s,\theta+2\pi_p)=J(\varrho,s-\omega(\theta+2\pi_p))-\int_0^\theta R(\varrho,s-\omega(\theta-\phi),\phi)d\phi+\Omega(\varrho,s-\omega(\theta+2\pi_p)). \tag{3.21}$$

Setting $S_3(\varrho,s,\theta+2\pi_p)=S_3(\varrho,s,\theta)$, it follows from (3.20) and (3.21) that

$$J(\varrho,s-\omega(\theta+2\pi_p))+\Omega(\varrho,s-\omega(\theta+2\pi_p))-\Omega(\varrho,s-\omega\theta)=0,$$

or equivalently,

$$J(\varrho, x) = \Omega(\varrho, x + x_0) - \Omega(\varrho, x), \tag{3.22}$$

where $x = s - \omega(\theta + 2\pi_p)$ and $x_0 = 2\omega\pi_p$.

From (3.18) and the definition of J, we have

$$\int_0^{2\pi_p} J(\varrho, x)dx = -\int_0^{2\pi_p}\int_0^{2\pi_p} R(\varrho, x + \omega\phi, \phi)dxd\phi = -\int_0^{2\omega\pi_p}\int_0^{2\pi_p} R(\varrho, x, \phi)dxd\phi = 0.$$

Thus we assume $J(\varrho, x) = \sum_{0\neq k\in Z} J_k(\varrho)e^{i\lambda kx}$ and $\Omega(\varrho, x) = \sum_{0\neq k\in Z}\Omega_k(\varrho)e^{i\lambda kx}$,where $\lambda = \pi/\pi_p$. Then the homological equation (3.22) implies that

$$\Omega_k = \frac{J_k}{e^{i\lambda kx_0} - 1}, \quad k \neq 0.$$

The definition of $J(\varrho, x)$ implies that $J(\varrho, x)$ is C^l on x. Thus it holds that

$$|J_k| \leq C \cdot \|J(\cdot, x)\|_{C^l} \cdot |k|^{-l}, \quad k \neq 0. \tag{3.23}$$

From the Diophantine condition (1.14), we have that

$$|e^{i\lambda kx_0} - 1| \geq 2\pi\gamma|k|^{-\tau}, \quad k \neq 0. \tag{3.24}$$

Combining (3.23) and (3.24), we obtain that

$$|\Omega_k| \leq C \cdot \|J(\cdot, x)\|_{C^l} \cdot |k|^{\tau - l}, \quad k \neq 0,$$

which implies Ω is well-defined and C^{l-3} on x since $1 < \tau < 2$.

For the definition of Ω and (3.14), we have that

$$|D_\varrho^i D_x^j \Omega| \leq C \cdot \varrho^{-\varepsilon - \frac{i}{q}}, \quad 0 \leq i + j \leq l - 2,$$

which together with (3.14) and (3.20) implies

$$|D_\varrho^i D_\tau^j S_3| \leq C \cdot \varrho^{-\varepsilon - \frac{i}{q}}, \quad 0 \leq i + j \leq l - 2. \tag{3.25}$$

Thus we obtain (3.16) from (3.17) and (3.25) and the proof is completed. □

By lemma 3.2 and the repeated use of lemma 3.3, we have the following result.

Corollary 3.1 *There exists a canonical transformation Φ_4 of the form:*

$$\Phi_4 : \quad \begin{cases} I &= \zeta + u_4(\zeta, \eta, \theta) \\ s &= \eta + v_4(\zeta, \eta, \theta) \end{cases}$$

such that the system with Hamiltonian (3.9) is transformed into the following one

$$\mathfrak{r}(\zeta, \eta, \theta) = \omega\zeta + \mathfrak{r}_1(\zeta) + \mathfrak{r}_2(\zeta, \eta, \theta), \tag{3.26}$$

where $\mathfrak{r}_1 = \bar{r}_1 + [\bar{r}_2]_0$ with $\bar{r}_1$, $[\bar{r}_2]_0$ satisfying (3.10), (3.12), and $\mathfrak{r}_2$ satisfies

$$|D_\zeta^i D_\eta^j \mathfrak{r}_2| \leq C \cdot \zeta^{-2-\frac{i}{q}} \tag{3.27}$$

for $0 \leq i + j \leq 5$.

4 Proof of theorem 1

In order to apply Moser's small twist theorem, we need to calculate the pontcareé mapping of the Hamiltonian system with the Hamiltonian (3.26). So in this section, we first give the expression of the Poincaré mapping. And then we will use Moser's small twist theorem to prove Theorem 1.

From corollary 3.1, it follows that the Hamiltonian system with the Hamiltonian (3.26) is of the form:

$$\begin{cases} \frac{d\eta}{d\theta} & = \omega + \mathfrak{r}_1'(\zeta) + \frac{\partial \mathfrak{r}_2}{\partial \zeta}(\zeta,\eta,\theta) \\ \frac{d\zeta}{d\theta} & = -\frac{\partial \mathfrak{r}_2}{\partial \eta}(\zeta,\eta,\theta), \end{cases} \tag{4.1}$$

where $\mathfrak{r}_1(\zeta) = \bar{r}_1(\zeta) + [\bar{r}_2]_0(\zeta)$ satisfying (3.10) and (3.12), $\mathfrak{r}_2(\zeta,\eta,\theta)$ satisfies (3.27).

Thus the Poincaré map of the equation (4.1) is of the form:

$$P: \begin{cases} \eta(2\pi_p) & = 2\pi_p\omega + \eta + \alpha(\zeta) + F_1(\zeta,\eta), \\ \zeta(2\pi_p) & = \zeta + F_2(\zeta,\eta). \end{cases} \tag{4.2}$$

where $F_1(\zeta,\eta) = \int_0^{2\pi_p} \frac{\partial \mathfrak{r}_2}{\partial \zeta}(\zeta,\eta,\theta)d\theta$, $F_2(\zeta,\eta) = -\int_0^{2\pi_p} \frac{\partial \mathfrak{r}_2}{\partial \eta}(\zeta,\eta,\theta)d\theta$,$\alpha(\zeta) = \mathfrak{r}_1'(\zeta)$, and from the definition of $\mathfrak{r}_1$, (3.10), (3.12) and (3.27), we have that

$$\alpha(\zeta) = \alpha_1(\zeta) + \alpha_2(\zeta) \tag{4.3}$$

with

$$\begin{array}{rl} & |\alpha_1^{(i)}(\zeta)| \geq c\cdot\zeta^{-\frac{1}{q}-i}, \\ |\alpha_1^{(i)}(\zeta)| \leq & C\cdot\zeta^{-\frac{1}{q}-i}, \quad |\alpha_2^{(i)}(\zeta)| \leq C\cdot\zeta^{-\delta_1-\frac{1}{q}-\frac{i}{q}}, \quad 0 \leq i \leq 4 \end{array} \tag{4.4}$$

and

$$|D_\zeta^i D_\eta^j F_k(\zeta,\eta)| \leq C\cdot\zeta^{-2-\frac{i}{q}}, \quad 0 \leq i+j \leq 4,\ k = 1,2, \tag{4.5}$$

where $\alpha_1(\zeta) = \bar{r}_1'(\zeta)$,$\alpha_2(\zeta) = [\bar{r}_2]_0'(\zeta)$.

According to (4.4), we can know that the following case is possible, that is, the function $\alpha(\zeta)$ may be not monotone. In order to find a monotone interval for $\alpha(\zeta)$, we consider the interval $[2\zeta_0, 3\zeta_0]$ with $\zeta_0 \gg 1$. By (4.3) and (4.4), we have that the set $\alpha([\frac{9}{4}\zeta_0, \frac{11}{4}\zeta_0])$ covers some interval with length longer than $c\cdot\zeta_0^{-\frac{1}{q}}$. Therefor by Mean Value theorem of Differentials, there exists some point $\zeta^* \in [\frac{9}{4}\zeta_0, \frac{11}{4}\zeta_0]$ such that $|\alpha'(\zeta^*)| \geq c\cdot\zeta_0^{-\frac{1+q}{q}}$.

What's more, (4.4) implies $|\alpha''(\zeta)| \leq C\cdot\zeta^{-\frac{1+q}{q}-\delta_1}$. Thus for each $\zeta \in [\zeta^*, \zeta^* + \zeta_0^{\frac{\delta_1}{q}}]$, we have

$$|\alpha'(\zeta)| \geq c\cdot\zeta_0^{-\frac{1+q}{q}}. \tag{4.6}$$

In the next, we give the following scale transformation :

$$\alpha(\zeta) - \alpha(\zeta^*) = \zeta_0^{-\frac{1+q}{q}}\nu, \quad \nu \in [2,3]. \tag{4.7}$$

Then we have the following Poincaré mapping:

$$\tilde{P}: \quad \begin{cases} \eta(2\pi_p) &= 2\pi_p\omega + \alpha(\zeta^*) + \eta + \zeta_0^{-\frac{1+q}{q}}\nu + \tilde{F}_1(\nu,\eta), \\ \nu(2\pi_p) &= \nu + \tilde{F}_2(\nu,\eta), \end{cases} \tag{4.8}$$

where

$$\tilde{F}_1(\nu,\eta) = F_1(\zeta(\nu),\eta), \quad \tilde{F}_2(\nu,\eta) = \zeta_0^{\frac{1+q}{q}}(\alpha(\zeta(\nu) + F_2(\zeta(\nu),\eta)) - \alpha(\zeta(\nu))) \tag{4.9}$$

with $\zeta(\nu)$ determined by (4.7).

From (4.4), (4.6) and (4.7), we see that

$$|\zeta^{(i)}(\nu)| \le C, \quad 1 \le i \le 4, \tag{4.10}$$

which together with (4.5) and (4.9) implies

$$|D_\nu^i D_\eta^j \tilde{F}_1| \le C \cdot \zeta_0^{-2}, \quad |D_\nu^i D_\eta^j \tilde{F}_2| \le C \cdot \zeta_0^{-2}, \quad 0 \le i+j \le 4. \tag{4.11}$$

What's more, the mapping $\tilde{P}$ of the Hamiltonian system (3.26) is time $2\pi_p$ mapping , so it is area-preserving. And further it possesses the intersection property in the annulus $[2,3] \times \mathbb{S}_p$, this is to say, if Γ is an embedded circle in $[2,3] \times \mathbb{S}_p$ homotopic to a circle $\nu =$ constant, then $\tilde{P}(\Gamma) \cap \Gamma \neq \emptyset$. The proof can be found in [4].

For the mapping $\tilde{P}$, all the conditions of Moser's small twist theorem [19] have been verified. Consequently, if $\zeta_0 \gg 1$, then there exists an invariant curve Γ of $\tilde{P}$ surrounding $\nu \equiv 1$. This implies that the Poincaré mapping of the system (3.26) indeed processes invariant curves. Retracting the sequence of transformations back to the original system, we conclude that there exist invariant curves of the Poincaré mapping of the original system (1.11). And those curves surround the origin $(x,y) = (0,0)$ and at the same time are arbitrarily far from it. This completes the proof of Theorem 1.

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